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GROUP ACTIONS ON A_k-MANIFOLDS

BY

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ABSTRACT. By an A_k-manifold we mean a connected manifold with elements $w_i \in H^1(M)$, $1 < i < k$, such that $w_1 \cup \cdots \cup w_k \neq 0$. In this paper we study the fixed point set, degree of symmetry, semisimple degree of symmetry and gaps of transformation groups on A_k-manifolds.

1. Introduction. A connected topological or differentiable m-manifold M^m is called an *A_k-manifold*, where k is a nonnegative integer, if there exist $w_i \in H^1(M; Q)$, $1 \leq i \leq k$, such that $w_1 \cup \cdots \cup w_k \neq 0$. Here $H^*(M; L)$ denotes the Alexander-Spanier cohomology with compact supports, and with coefficients in L. It follows from the definition that any connected manifold is an A_0-manifold. For example, the connected sum $(T^k \times S^{m-k}) \# M^m$ is an A_k-manifold.

Let M be a topological m-manifold. The *degree of symmetry* $N_T(M)$ (resp. *semisimple degree of symmetry* $N_T^s(M)$) of M is defined as the supremum of the dimensions of all compact (resp. compact semisimple) Lie groups which can act effectively on M. If M is a differentiable manifold, the *degree of symmetry* $N(M)$ [9] and *semisimple degree of symmetry* $N^s(M)$ can be similarly defined by assuming the actions to be differentiable. It is easy to verify that $N_T(M^m) \leq m + N_T^s(M)$ (resp. $N(M^m) \leq m + N^s(M^m)$), and if $N_T(M^m) = m + N_T^s(M^m)$ (resp. $N(M^m) = m + N^s(M^m)$), then M^m is homeomorphic (resp. diffeomorphic) to the m-torus T^m [5]. Moreover, there is an interesting connection with the differential geometry, that is, if $N^s(M^m) \neq 0$, then M admits a Riemannian metric with positive scalar curvature [18].

Now let G be a compact Lie group and $G \to E_G \to B_G$ a universal G-bundle. For a G-space X, the equivariant cohomology of X with coefficients in L is defined by

$$H_G^*(X; L) = H^*(E_G \times_G X; L).$$

We shall omit the coefficients L if $L = Z$ or Q in most cases.

In this paper we shall investigate the transformation groups on A_k-manifolds. First, we examine the fixed point sets of A_k-manifolds via the index

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homomorphism and prove the following result.

THEOREM A. *Let the torus group $G = T^n$ act effectively and differentiably on a compact connected A_k-manifold M. Suppose $K(M)$ is a polynomial in the Pontrjagin classes of M with rational coefficients such that*

$$\langle w_1 \cup \cdots \cup w_k \cup K(M), [M] \rangle \neq 0,$$

where $[M]$ denote the fundamental class of M. If the fixed point set F is not empty, then at least one component of F is also an A_k-manifold.

A well-known classical result in Riemannian geometry states that if M is a compact closed differentiable m-manifold, then $N(M^m) \leqslant \langle m \rangle$, and that $N(M^m) = \langle m \rangle$ if and only if M is diffeomorphic to either the standard sphere S^m or the standard real projective space RP^m [7], where $\langle n \rangle$ denotes $n(n+1)/2$ for a nonnegative integer n. We generalize this result by proving the following:

THEOREM B. *Suppose M^m is an m-dimensional A_k-manifold. Then*

(i) $N_T^s(M) \leqslant \langle m - k \rangle$.

(ii) $N_T(M) \leqslant k + \langle m - k \rangle$.

(iii) *Suppose M is compact. Then $N_T(M) = k + \langle m - k \rangle$ if and only if M is homeomorphic to $S^{m-k} \times T^k$, $RP^{m-k} \times T^k$ or $S^{m-k} \times_{Z_2} T^k$.*

In the differentiable category, (i) is proved by Yau in [23]. For $m = k$, Theorem B is also verified by Burghelea and Schultz [5, Theorem A]. If, in addition, we assume that $H^\alpha(M; Q) \neq 0$, we can establish a sharper bound for the invariants $N_T^s(M)$ and $N_T(M)$. More precisely, we have

THEOREM C. *Suppose M^m is an A_k-manifold, $m - k > 19$. If there exists an element $u \in H^\alpha(M; Q)$ such that*

$$w_1 \cup \cdots \cup w_k \cup u \neq 0,$$

then precisely one of the following holds:

(i) $N_T^s(M) \leqslant \langle \bar{m} - \alpha \rangle + \langle \alpha \rangle$,

(ii) $N_T^s(M) = \dim SU(\bar{m}/2 + 1)$, *where* $\bar{m} = m - k$.

THEOREM D. *Suppose M^m is an A_k-manifold. If there exists an element $u \in H^\alpha(M; Q)$ such that*

$$w_1 \cup \cdots \cup w_k \cup u \neq 0, \qquad \bar{m} > 19 + \alpha,$$

then one of the following holds:

(i) $N_T(M) \leqslant k + \langle \bar{m} - \alpha \rangle + \langle \alpha \rangle$,

(ii) $N_T(M) \leqslant k + \dim SU(\bar{m}/2 + 1)$.

Theorem D generalizes Theorem 1 in [16].

Now for a positive integer n, let $\Phi(n)$ denote the largest integer j satisfying the inequality[1]

$$\langle n-j\rangle + \langle j\rangle < \langle n-j+1\rangle.$$

It is evident that

$$n > \langle i\rangle + i - 1 \text{ and } \langle n-i\rangle \geqslant \langle n-\Phi(n)\rangle > n^2/4 + n,$$
$$\text{for } i = 1, 2, \ldots, \Phi(n) \text{ if } n \geqslant 17. \tag{1}$$

Let m, k be positive integers, $m > k$ and m_i, $1 \leqslant i \leqslant s+2$, a sequence of nonnegative integers satisfying the following conditions:

$$\begin{aligned} &m_i = k_{i-1} - k_i,\ 1 \leqslant i \leqslant s+1 \text{ and } m_s \geqslant m_{s+2} = k_{s+1}, \text{ where}\\ &k_i,\ 0 \leqslant i \leqslant s+1, \text{ is a sequence of nonnegative integers with}\\ &k_0 = m-k,\ k_{i+1} \leqslant \Phi(k_i),\ 0 \leqslant i \leqslant s-1 \text{ if } s \geqslant 1. \end{aligned} \tag{*}$$

The following theorem generalizes Theorem C.

THEOREM E. *Let M^m be a topological A_k-manifold, and there exist $x_i \in H^{m_i}(M; Q)$ for $1 \leqslant i \leqslant s$ and $x_{s+1} \in H^{m_{s+1}+j}(M; Q)$, $j = 1$ or 2, such that*

$$\prod_{i=1}^{k} w_i \cup \prod_{j=1}^{s+1} x_j \neq 0,$$

where the sequence m_i, $1 \leqslant i \leqslant s+2$, satisfies (*) and $k_1 \geqslant 19$. Then one of the following holds:

(i) $N_T^s(M) \leqslant \sum_{i=1}^{s+2}\langle m_i\rangle$,

(ii) $N_T^s(M) = \sum_{i=1}^{s}\langle m_i\rangle + \dim SU((m_{s+1}+m_{s+2})/2+1)$.

We can improve Theorem D in the smooth category and prove

THEOREM F. *Let M^m be a compact connected differentiable A_k-manifold, and $x_i \in H^{m_i}(M; Q)$, $1 \leqslant i \leqslant s+2$, such that*

$$\prod_{i=1}^{k} w_i \cup \prod_{j=1}^{s+2} x_j \neq 0,$$

where the sequence m_i satisfies (*) and $k_s \geqslant 19 + m_{s+2}$. Then one of the following holds:

(i) $N(M) \leqslant k + \sum_{i=1}^{s+2}\langle m_i\rangle$,

(ii) $N(M) \leqslant k + \sum_{i=1}^{s}\langle m_i\rangle + \dim SU((m_{s+1}+m_{s+2})/2+1)$.

Finally, we prove the following generalized gaps theorem.

THEOREM G. *Let M^m be a connected topological m-dimensional A_k-manifold, and k_i, $1 \leqslant i \leqslant s+1$, a sequence of nonnegative integers with $k_0 = m-k$, $k_{i+1} \leqslant \Phi(k_i)$, $0 \leqslant i \leqslant s$, and $k_s \geqslant 17$. Suppose G is a compact connected Lie*

[1] This definition of $\Phi(n)$ is a slight improvement of Mann's definition in [20] which is defined as $\sup\{j \mid \langle n-j\rangle + \langle j\rangle < \langle n-j+1\rangle - 1\}$.

group acting effectively on M, and $q \geqslant k$, where we express G as $(T^q \times K)/N$, K simply connected semisimple and N a finite normal subgroup of $T^q \times K$. Then the dimension of G cannot fall into any of the following ranges:

$$k + \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} \rangle + \langle k_{s+1} \rangle$$
$$< \dim G < k + \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} + 1 \rangle.$$

For $k = 0$, Theorem F is the "further gaps theorem" of Mann **[20]**.

2. Index homomorphism and the fixed point set. Throughout this section we assume that all manifolds are compact connected orientable and differentiable and the actions are smooth.

Let M and N be differentiable G-manifolds and f: $M \to N$ be a differentiable equivariant G-map. Choose an equivariant embedding e: $M \to V$, where V is a complex linear G-space. Let ν be the normal bundle of the embedding

$$\tilde{f} = f \times e \colon M \to N \times V.$$

Denote the disk and sphere bundle (resp. unit disk and unit sphere) of ν (resp. V) by $D(\nu)$ and $S(\nu)$ (resp. D and S). We can assume that $\tilde{f}$: $M \to N \times D$. The *Gysin homomorphism*

$$f_!\colon H_G^*(M) \to H_G^*(N)$$

is defined as $f_! = \phi_0^{-1}\bar{f}^*\phi$

$$H_G^*(M) \xrightarrow{\phi} H_G^*(D(\nu)/S(\nu)) \xrightarrow{\bar{f}^*} H_G^*(N \times D/N \times S) \xrightarrow{\phi_0^{-1}} H_G^*(N)$$

where ϕ and ϕ_0 are Thom isomorphisms and $\bar{f}^*$ is induced by the collapsing map

$$\bar{f}\colon N \times D/N \times D \to D(\nu)/S(\nu).$$

If N is a point, we denote the Gysin homomorphism by

$$\text{Ind}\colon H_G^*(M) \to H^*(B_G),$$

and it is called the *index homomorphism* **[11]**, **[12]**, **[22]**.

For a vector bundle ξ, denote $\chi(\xi)$ its Euler class. We shall denote the vector bundle

$$R^n \to E_G \times_G R^n \to B_G$$

simply by $E_G \times_G R^n$, where G acts orthogonally on R^n. Now let $G = T^n$ and S be the multiplicative subset of $H^*(B_G)$ defined by

$$S = \{\chi(E_G \times_G R^n) | G \text{ acts on } R^n \text{ without trivial direct summand}\}.$$

If $G = S^1$, then $S = \Sigma_{i>0} H^{2i}(B_G)$ and $H^*(B_G) = Z[t]$, $\deg t = 2$. Let F be the fixed point set of the action of G on M. Then there is a localization isomorphism [**10**], [**24**]

$$S^{-1}i^*\colon S^{-1}H_G^*(M) \cong S^{-1}H_G^*(F) = S^{-1}H^*(B_G \times F),$$

where $i\colon F \to M$ is the inclusion. Let $F = \bigcup_{j=1}^{s} F_j$, F_j components, and $i_j\colon F_j \to M$ the inclusion with normal bundle ν_j. Denote the Euler class of the bundle

$$E_G \times_G \nu_j \to E_G \times_G F_j$$

simply by $\chi_G(\nu_j)$ and call it the equivariant Euler class of ν_j. It is known that (cf. [**11**])

$$i_j^* i_{j!}(x) = \chi_G(\nu_j) \cdot x$$

for $x \in H_G^*(F_j)$ and $\chi_G(\nu_j)$ is a unit in $S^{-1}H_G^*(F_j)$.

THEOREM 2.1 (LEFSCHETZ FIXPOINT FORMULA [**11**], [**22**]). *Let $G = T^n$ act on a manifold M with fixed point set $F = \bigcup_{j=1}^{s} F_j$. Then the following diagram commutes.*

$$\begin{array}{ccc}
S^{-1}H_G^*(M) & \xrightarrow{\ S^{-1}\mathrm{Ind}\ } & S^{-1}H^*(B_G) \\
\Big\downarrow {\scriptstyle S^{-1}\sum_j i_j^*/\chi_G(\nu_j)} & \nearrow {\scriptstyle \sum_j S^{-1}\mathrm{Ind}_j} & \\
\sum_j S^{-1}H_G^*(F_j) & &
\end{array}$$

Since the natural map $H^*(B_G) \to S^{-1}H^*(B_G)$ is injective, for any $u \in H_G^*(M)$ we have

$$\mathrm{Ind}\, u = S^{-1}\sum_j \mathrm{Ind}\, i_j^*(u)/\chi_G(\nu_j),$$

in $S^{-1}H^*(B_G)$.

THEOREM 2.2. *Suppose $G = T^n$ acts on a manifold M^m. Let $a_i \in H^{n_i}(M)$ and $b_i \in H_G^{n_i}(M)$, $1 < i < k$, be such that*

(i) $\langle a_1 \cup \cdots \cup a_k, [M]\rangle \neq 0$,

(ii) $i^*b_i = a_i$, $1 < i < k$, *where $i^*\colon H_G^*(M) \to H^*(M)$ is induced by the inclusion $i\colon M \to E_G \times_G M$.*

Then the fixed point set F is not empty.

PROOF. There is a commutative diagram [**12**]

$$\begin{array}{ccc}
H_G^*(M) & \xrightarrow{\quad \mathrm{Ind} \quad} & H^*(B_G) \\
\Big\downarrow {\scriptstyle j_m^*} & & \Big\downarrow {\scriptstyle j_m^*} \\
H^*(S^{2m+1} \times \cdots \times S^{2m+1} \times_G M) & \xrightarrow{\ P_!\ } & H^*(CP^m \times \cdots \times CP^m),
\end{array}$$

where $P_!$ is the Gysin map (the Poincaré dual of the homology homomorphism), and j_m^* is induced by inclusion. Take $m = 0$; then

$$\begin{aligned} j_0^*\operatorname{Ind}(b_1 \cup \cdots \cup b_k) &= P_! j_0^*(b_1 \cup \cdots \cup b_k) = P_! i^*(b_1 \cup \cdots \cup b_k) \\ &= P_!(a_1 \cup \cdots \cup a_k) = \langle a_1 \cup \cdots \cup a_k, [M] \rangle \neq 0. \end{aligned}$$

Hence, $\operatorname{Ind}(b_1 \cup \cdots \cup b_k) \neq 0$. Notice that

$$P_!: H^m(S^1 \times \cdots \times S^1 \times_G M) = H^m(M) \to H^0(CP^0 \times \cdots \times CP^0)$$

and

$$j_0^*: H^0(B_G) \to H^0(CP^0 \times \cdots \times CP^0)$$

can be identified as the identity maps. On the other hand, it follows from the Lefschetz fixpoint formula that

$$\operatorname{Ind}(b_1 \cup \cdots \cup b_k) = S^{-1} \sum_j \operatorname{Ind} i_j^*(b_1 \cup \cdots \cup b_k)/\chi_G(\nu_j) \neq 0.$$

Therefore, F is not empty.

This simple result includes many known results in smooth transformation groups concerning the existence of the fixed point set. For instance:

(a) M is totally nonhomologous to zero in the fibration $M \to E_G \times_G M \to B_G$; then i^* is surjective (cf. [3]).

(b) A Pontrjagin number of M is nonzero (cf. [6]).

(c) The Euler characteristic $\chi(M) \neq 0$. This can be proved as follows. Since we have

$$i^*(\chi(E_G \times_G M)) = \chi(TM),$$

and

$$i_j^*(\chi(E_G \times_G M)) = \chi(B_G \times_G TF_j)\chi_G(\nu_j) = \chi(TF_j)\chi_G(\nu_j)$$

(where TM denotes the tangent bundle of M),

$$\begin{aligned} \chi(M) &= \operatorname{Ind} \chi(E_G \times_G TM) \\ &= \sum_j \operatorname{Ind} \chi i_j^*(E_G \times_G TM)/\chi_G(\nu_j) \\ &= \sum_j \operatorname{Ind} \chi(TF_j) = \sum_j \chi(F_j) = \chi(F). \end{aligned}$$

LEMMA 2.3. *Let T^n act almost effectively on the manifold M, and*

$$i^*: H^i_{T^n}(M; Q) \to H^i(M; Q)$$

is induced by inclusion.

(i) *If F is not empty, then i^* is surjective for $i = 1$.*

(ii) *Let T^n be a maximal torus of a compact connected semisimple Lie group G, and the action of T^n on M extends to an action of G. Then i^* is surjective for $i = 1, 2$.*

PROOF. (i). Let π: $M \to M/T^n$ be the orbit projection. By [4, p. 161]

$$\pi^*: H^1(M/T^n; Q) \to H^1(M; Q)$$

is surjective. The result follows from the following commutative diagram:

$$\begin{array}{ccc} H^1(M/T^n; Q) & \xrightarrow{\pi^*} & H^1(M; Q) \\ \downarrow \pi_1^* & \nearrow i^* & \\ H^1_{T^n}(M; Q) & & \end{array}$$

where π_1: $E_{T^n} \times_{T^n} M \to M/T^n$ is the projection.

(ii) Since G is semisimple,

$$H^i(B_G; Q) = 0, \qquad i = 1, 2, 3.$$

Hence, it follows from the spectral sequence of the fibration

$$M \to E_G \times_G M \to B_G$$

that $\bar{i}^*$ is onto for $i = 1, 2$, where $\bar{i}^*$: $H^i_G(M; Q) \to H^i(M; Q)$. Hence the required conclusion follows from the following commutative diagram, where j^* is induced by the inclusion j: $T^n \to G$.

$$\begin{array}{ccc} H^*_G(M; Q) & \searrow \bar{i}^* & \\ \downarrow j^* & & H^*(M; Q) \\ H^*_{T^n}(M; Q) & \nearrow i^* & \end{array}$$

PROOF OF THEOREM A. The proof is almost identical with the proof of Theorem 2.2. By naturality of Pontrjagin classes, if we let $K_G(M) = K(E_G \times_G TM)$ be a polynomial in the Pontrjagin classes of $E_G \times_G TM$ by using the same polynomial as the one used in $K(M)$, then $i^*K_G(M) = K(M)$. By Lemma 2.4, i^*: $H^1_G(M; Q) \to H^1(M, Q)$ is surjective; hence there exists $\bar{w}_i \in H^1_G(M; Q)$, $1 \leq i \leq k$, and

$$j_0^*\mathrm{Ind}(\bar{w}_1 \cup \cdots \cup \bar{w}_k \cup K_G(M)) = \langle w_1 \cup \cdots \cup w_k \cup K(M), [M] \rangle \neq 0.$$

Again, by the Lefschetz fixed point formula we have

$$\begin{aligned} \mathrm{Ind}(\bar{w}_1 \cup \cdots \cup \bar{w}_k \cup K_G(M)) &= S^{-1}\sum_j \mathrm{Ind}\, i_j^*(\bar{w}_1) \cdots i_j^*(\bar{w}_k) i_j^*(K_G(M))/\chi_G(\nu_j) \\ &= \sum_j \{ i_j^*(\bar{w}_1) \cdots i_j^*(\bar{w}_k) K(E_G \times_G (TF_j \oplus \nu_j))/\chi_G(\nu_j)\}/[F_j], \end{aligned}$$

where $/[F_j]$ denotes the slant product, and

$$i_j^*(\bar{w}_i) \in H^1(F_j; Q) = H^1(B_G \times F_j; Q).$$

Hence, there exists at least one j such that

$$0 \neq i_j^*(\bar{w}_1) \cdots i_j^*(\bar{w}_k) \in H^*(F_j; Q).$$

This completes the proof of Theorem A.

Let T^n be a maximal torus of a compact connected semisimple Lie group G, and G acts effectively on the A_k-manifold M. By Lemma 2.3 there exists $\bar{w}_i \in H^1_{T^n}(M; Q)$ such that $i^*(\bar{w}_i) = w_i$, $1 \leq i \leq k$. We have the commutative diagram

$$\begin{array}{ccc} H_G^*(M; Q) & \xrightarrow{\text{Ind}} & H^*(B_G; Q) \\ j^*\downarrow & & \downarrow j^* \\ H_{T^n}^*(M; Q) & \xrightarrow{\text{Ind}} & H^*(B_{T^n}; Q) \end{array}$$

where j^* is induced by the inclusion. It is well known that

$$j^* H^*(B_G; Q) = H^*(B_{T^n}; Q)^{W(G)},$$

where $W(G)$ denotes the Weyl group of G. By an easy spectral sequence argument there exist $\tilde{w}_j \in H^1_G(M; Q)$ such that $j^*\tilde{w}_j = \bar{w}_j$, $1 \leq j \leq k$. Hence

$$\begin{aligned} &\text{Ind}(\bar{w}_1 \cup \cdots \cup \bar{w}_k \cup K_{T^n}(M)) \\ &\qquad = j^*\text{Ind}(\tilde{w}_1 \cup \cdots \cup \tilde{w}_k \cup K_G(M)) \in H^*(B_{T^n}; Q)^{W(G)}. \end{aligned}$$

Thus we have proved

PROPOSITION 2.4. *Let T^n be a maximal torus subgroup of a compact connected semisimple Lie group which acts effectively on the A_k-manifold M. If T^n is extendable to an action of G on M, then*

$$\text{Ind}(\bar{w}_1 \cup \cdots \cup \bar{w}_k \cup K_{T^n}(M)) \in H^*(B_{T^n}; Q)^{W(G)}.$$

In particular, if $G = S^3$, then for $S^1 \subset S^3$

$$\text{Ind}(\bar{w}_1 \cup \cdots \cup \bar{w}_k \cup K_{S^1}(M)) \in Q[t^2].$$

EXAMPLES. Let $M = T^k \times CP^3$, $k > 0$. Define an action of S^1 on M as follows. For $g \in S^1$, $(x, [z_0, z_1, z_2, z_3]) \in T^k \times CP^3$, define

$$g(x, [z_0, z_1, z_2, z_3]) = (x, [g^{a_0}z_0, g^{a_1}z_1, g^{a_2}z_2, g^{a_3}z_3])$$

where a_0, a_1, a_2 and a_3 are distinct integers. We have fixed point set $F = \cup_{j=1}^4 F_j$ with

$$F_j = T^k \times \{x_j\}, \quad x_0 = [1, 0, 0, 0], \ldots, \quad x_3 = [0, 0, 0, 1].$$

The local weights $\Omega(F_j)$ of the representation of S^1 on ν_j restrict to a point in F_j is given by

$$\Omega(F_j) = \{(a_i - a_j)t \mid i \neq j\}.$$

Hence

$$P(E_{S^1} \times_{S^1} \nu_j) = \prod_{i \neq j} \{1 + (a_i - a_j)^2 t^2\},$$

and

$$\chi_{S^1}(\nu_j) = \prod_{i \neq j} (a_i - a_j) t^3,$$

where $P(E_{S^1} \times_{S^1} \nu_j)$ denotes the total Pontrjagin class of the bundle $E_{S^1} \times_{S^1} \nu_j \to B_{S^1} \times F_j$. Thus if we let $\bar{w}_j \in H^1_{S^1}(M)$ be such that $i^* \bar{w}_j = w_j$, $\{w_1, \ldots, w_k\}$ a base of $H^1(M, Q)$ such that $\langle w_1 \cup \cdots \cup w_k, [T^k] \rangle = 1$, then

$$\begin{aligned}
&\operatorname{Ind} \bar{w}_1 \cdots \bar{w}_k P(E_{S^1} \times_{S^1} TM) \\
&= S^{-1} \sum_j \operatorname{Ind} \bar{w}_1 \cdots \bar{w}_k P(E_{S^1} \times_{S^1} TM)/\chi(E_{S^1} \times_{S^1} \nu_j) \\
&= \sum_j P(E_{S^1} \times_{S^1} \nu_j)/\chi(E_{S^1} \times_{S^1} \nu_j) \\
&= \sum_j \left\{ \prod_{i \neq j} \{1 + (a_i - a_j)^2 t^2\} / \prod_{i \neq j} (a_i - a_j) t^3 \right\} \\
&= (a_3 + a_2 - a_1 - a_0)(a_1 + a_2 - a_0 - a_3)(a_1 + a_3 - a_0 - a_2) t^3 \notin Q[t^2]
\end{aligned}$$

if $a_0 + a_1 \neq a_2 + a_3$, $a_0 + a_2 \neq a_1 + a_3$ and $a_0 + a_3 \neq a_1 + a_2$. By Proposition 2.4 these actions cannot extend to the action of S^3.

By using the same technique, it is easy to construct T^2 actions on $T^k \times CP^3$ which are not extendable to actions of SU(3).

3. Degree of symmetry and semisimple degree of symmetry. The following lemma may be found in [14], [15].

LEMMA 3.1. *Let $G = G_1 \times K$ be a compact connected Lie group acting effectively on a connected topological manifold M. Suppose $\dim G_1 = N_T(G_1(x))$, where $G_1(x)$ is a principal G_1 orbit in M. Then K acts almost effectively on the orbit space M/G_1.*

We shall always express a compact connected Lie group in the following form, and call the G_i's the *normal factors* of G (or $\bar{G}$):

$$G = \bar{G}/N = (T^q \times K)/N = (T^q \times G_1 \times \cdots \times G_v)/N, \qquad (2)$$

where T^q is a q-torus ($q \geq 0$), each G_i is either simple simply connected or isomorphic to Spin(4) $\cong$ Spin(3) $\times$ Spin(3), and there is at most one Spin(3), and N is a finite normal subgroup of $\bar{G}$. Note that the group K is semisimple.

LEMMA 3.2 [19]. *Let G be a compact connected Lie group acting almost* *effectively on a connected topological manifold M, and t denote the dimension of a principal orbit. If G has a decomposition of the form* (2), *then there exist positive integers* t_i *such that* $\dim G_i \leqslant \langle t_i \rangle$, $1 \leqslant i \leqslant v$, *and* $\sum_{i=1}^{v} t_i \leqslant t - q$.

As an easy consequence of Lemma 3.2, we have

COROLLARY 3.3 [14], [15]. *Suppose the compact connected Lie group G acts almost effectively on a connected topological m-manifold M. Let the positive integers* t_j *satisfy*

(i) $\dim G_i \leqslant \langle t_i \rangle$, $1 \leqslant i \leqslant v$, *and* $\sum_{i=1}^{v} t_i \leqslant m - q$,

(ii) $t_j \leqslant t_1 \leqslant \beta \leqslant m$, $2 \leqslant j \leqslant v$, *for some integer* β.

Then

$$\dim G \leqslant \langle \beta \rangle + \langle m - \beta \rangle.$$

LEMMA 3.4 [16, MAIN LEMMA]. *Let G be a compact connected Lie group acting effectively on a connected topological m-manifold M,* $m \geqslant 19$. *Then if*

$$\dim G \geqslant m^2/4 + m/2,$$

exactly one of the following holds:

(α) *M is diffeomorphic to* CP^k ($m = 2k$), *G acts transitively on M and G is locally isomorphic to* $SU(k+1)$.

(β) *M is diffeomorphic to* $CP^k \times S^1$ ($m = 2k+1$), *G acts transitively on M and G is locally isomorphic to* $U(k+1)$.

(γ) *M is a simple lens space finitely covered by* S^{2k+1} ($m = 2k+1$), *G acts transitively on M and G is locally isomorphic to* $U(k+1)$.

(δ) *G contains a normal factor* $G_1 \approx \mathrm{Spin}(n)$ *where*

(a) $n \geqslant m/2 + 1$,

(b) G_1 *acts almost effectively on M with principal isotropy subgroup H whose identity component* H^0 *is a standard imbedded* $\mathrm{Spin}(n-1)$ *in* $\mathrm{Spin}(n)$.

The assumption of compactness in the Main Lemma of [16] is unnecessary as we have observed in [14].

LEMMA 3.5. *Let* k_0, k_1 *be positive integers satisfying* $\Phi(k_0) \geqslant k_1 \geqslant 17$. *Then*

$$\langle k_0 - k_1 - u \rangle + \langle k_1 + u \rangle \leqslant \langle k_0 - k_1 \rangle + \langle k_1 - \Phi(k_1) \rangle \tag{3}$$

if $1 \leqslant u < k_0/2 - k_1 - 1$.

PROOF. To verify (3), it is enough to prove the following inequality holds.

$$u(2k_0 - 4k_1 - 2u) \geqslant 2k_1\Phi(k_1) - \Phi(k_1)^2 + \Phi(k_1).$$

Let $f(u) = u(2k_0 - 4k_1 - 2u) - \{2k_1\Phi(k_1) - \Phi(k_1)^2 + \Phi(k_1)\}$. Then $f'(u) > 0$ if $u < k_0/2 - k_1$. Hence $f(u)$ is increasing for $1 \leqslant u < k_0/2 - k_1 - 1$. We see easily that $f(1) \geqslant 0$. This proves (3).

We shall always assume that G is a compact connected Lie group acting effectively on M with $\dim G = N_T(M)$ (resp. $\dim G = N_T^s(M)$, $N(M)$ or $N^s(M)$ depending on the hypothesis). In order to get a better estimate, we shall assume that k is the largest integer such that $w_1 \cup \cdots \cup w_k \neq 0$ in defining the A_k-manifold.

PROOF OF THEOREM B. (i) Let $\pi: M \to M/G$ be the orbit projection. Since the group G is semisimple, $H^1(G(x); Q) = 0$ for all $x \in M$. Hence,

$$\pi^*: H^1(M/G; Q) \to H^1(M; Q)$$

is an isomorphism by the Vietoris-Begle mapping theorem and $\dim M/G \geqslant k$ because M is an A_k-manifold. On the other hand, if we let $G(x)$ be a principal orbit, then $\dim M/G = m - \dim G(x)$ [3]. It follows that $\dim G(x) \leqslant m - k$, and $\dim G \leqslant \langle m - k \rangle$.

(ii) Since $N_T(T^m) = m$, we may assume that M is not homeomorphic to T^m. Express the group G in the form (2). Then $\dim K \leqslant \langle m - k \rangle$ by (i). We consider two cases.

(a) $\dim K = \langle m - k \rangle$. Then the group K must act almost effectively on a principal K-orbit of dimension $m - k$, K isomorphic to $\mathrm{Spin}(m - k + 1)$ and $K(x) \approx S^{m-k}$ or RP^{m-k}. It is well known that $N_T(K(x)) = \dim K$; hence T^q acts almost effectively on the orbit space M/K by Lemma 3.1. Note that $\dim M/K = k$. It follows that $q \leqslant k$ and hence

$$N_T(M) = \dim G = q + \dim K \leqslant k + \langle m - k \rangle.$$

(b) $\dim K < \langle m - k \rangle$. Apply the gap theorem [19] to the manifold W^{m-k}, where

$$W = \begin{cases} K(x) & \text{if } \dim K(x) = m - k, \\ K(x) \times S^{m-k-\dim K(x)} & \text{with } K \text{ acting trivially on } S^{m-k-\dim K(x)} \\ & \text{if } \dim K(x) < m - k. \end{cases} \tag{4}$$

Then we have

$$\dim K \leqslant \langle m - k - 1 \rangle + 1 \tag{5}$$

with the following three exceptions:

$$m - k = 4, \quad W \approx CP^2, \quad K \approx \mathrm{SU}(3), \tag{6}$$

$$m - k = 6, \quad W \approx S^6, \quad K \approx G_2,$$

$$\text{the exceptional simple Lie group of rank 2,} \tag{7}$$

$$m - k = 10, \quad W \approx CP^5, \quad K \approx \mathrm{SU}(6). \tag{8}$$

Suppose $N_T(M) > k + \langle m - k \rangle$, we will proceed to show that all cases (5) through (8) are impossible. Suppose (5) holds. Then $q > k + \langle m - k \rangle - \dim K \geqslant m - 1$. Or $q \geqslant m$. Hence M is homeomorphic to T^m which is a contradiction. Now rank $G \leqslant m$. But if any of (6), (7) and (8) holds, it would imply that rank $G \geqslant m + 1$. For example, let us assume (6); then $q > k + \langle 4 \rangle - 8 = k + 2$, and rank $G = q + 2 \geqslant k + 5 \geqslant m + 1$.

(iii) By hypothesis $\dim G = N_T(M) = k + \langle m - k\rangle$. Then

$$k + \langle m - k\rangle = \dim G = q + \dim K \leqslant q + \langle m - k\rangle.$$

Hence $k \leqslant q$. We will show that $k < q$ cannot occur. If not, $m - q \leqslant m - k - 1$. The group K acts almost effectively on M/T^q by Lemma 3.1. Hence

$$\dim K \leqslant \langle m - q\rangle \leqslant \langle m - k - 1\rangle = \langle m - k\rangle - m + k$$

and

$$k + \langle m - k\rangle = \dim G \leqslant q + \langle m - k\rangle - m + k.$$

This implies that $m = k = q$ which is an obvious contradiction. Hence we have $k = q$, $K \approx \mathrm{Spin}(m - k + 1)$ and K acts transitively on $M/T^q \approx S^{m-k}$ or RP^{m-k}. Again, by Lemma 3.1, T^k acts almost effectively and transitively on M/K; hence M/K is homeomorphic to T^k.

Notice that the action of K on M has all orbits of the same type. This follows from the fact that any point in M can be expressed as gtx for $g \in K$, $t \in T^k$, and a fixed $x \in M$. Moreover, $K_{gtx} = gK_xg^{-1}$. Hence, we have a fibre bundle

$$K/K_x \to M \to M/K \approx T^k$$

with associated principal bundle

$$L \to F(K_x, M) \to T^k$$

where $F(K_x, M)$ denotes the fixed point set of K_x action on M, $L = N(K_x, \mathrm{Spin}(m - k + 1))/K_x$, the normalizer of K_x in $\mathrm{Spin}(m - k + 1)$ and $K_x = N(\mathrm{Spin}(m - k), \mathrm{Spin}(m - k + 1))$, or $\mathrm{Spin}(m - k)$. Thus $L =$ identity or Z_2 and

$$M \approx K/K_x \times_L F(K_x, M)$$

by [17]. The result follows easily.

Proof of Theorem C. Let

$$N_T^s(M) = \dim G > \langle \bar{m} - \alpha\rangle + \langle \alpha\rangle. \tag{9}$$

We shall proceed to show that (ii) holds. From (9) we have

$$N_T^s(M) \geqslant \bar{m}^2/4 + \bar{m}/2.$$

By Theorem B, for any principal orbit, say $G(x)$, $\dim G(x) \leqslant \bar{m}$. Define W as in the proof of Theorem A (using G instead of K). Apply Lemma 3.4 to the action of G on W, we have the following two possibilities:

(α) The principal orbit $G(x) = W \approx CP^{\bar{k}}$, $\bar{m} = 2\bar{k}$ and

$$\dim G = \dim \mathrm{SU}(\bar{k} + 1) = \dim \mathrm{SU}(\bar{m}/2 + 1).$$

This implies (ii).

(δ) G contains a normal factor $G_1 \approx \mathrm{Spin}(n)$, $n \geqslant \bar{m}/2 + 1$, and G_1 acts almost effectively on M with orbits some combination of fixed points, RP^{n-1} and S^{n-1}. Let $\beta = \max(\alpha, \bar{m} - \alpha)$. Then we can show that $n - 1 = t_1 \leqslant \beta$;

hence $\beta \geqslant t_1 \geqslant t_j$, $2 \leqslant j \leqslant v$. By Corollary 3.3,

$$\dim G \leqslant \langle \bar{m} - \beta \rangle + \langle \beta \rangle = \langle \bar{m} - \alpha \rangle + \langle \alpha \rangle,$$

which contradicts (9).

PROOF OF THEOREM D. Assume $\dim G = N_T(M)$. If $q \leqslant k$, the result follows immediately from Theorem C. So we shall assume that $q > k$. Suppose

$$N_T(M) > k + \langle \bar{m} - \alpha \rangle + \langle \alpha \rangle.$$

Recall that $\beta = \max(\alpha, \bar{m} - \alpha)$.

(a) $m + 1 \leqslant q + \beta$, that is, $m - q \leqslant \beta - 1$. Since $\dim K \leqslant \langle m - q \rangle$, we have

$$\dim G \leqslant q + \langle m - q \rangle \leqslant q + \langle \beta - 1 \rangle \leqslant k + \langle \beta \rangle + \langle \bar{m} - \beta \rangle.$$

(b) $m \geqslant q + \beta$. Then $m - q \geqslant \beta \geqslant 19$ by hypothesis and

$$(q - k)(q + k + 1 + 2\alpha - 2m) \leqslant 0. \tag{10}$$

Suppose $\dim K \leqslant \langle m - q - \alpha \rangle + \langle \alpha \rangle$. Then

$$\dim G \leqslant q + \langle m - q - \alpha \rangle + \langle \alpha \rangle \leqslant k + \langle \bar{m} - \alpha \rangle + \langle \alpha \rangle \quad \text{(by (10))},$$

which is a contradiction. Hence,

$$\dim K > \langle m - q - \alpha \rangle + \langle \alpha \rangle \geqslant \tilde{m}^2/4 + \tilde{m}/2,$$

where $\tilde{m} = m - q \geqslant 19$. Now we can repeat the argument of the proof of Theorem C to obtain either

$$\dim G = q + \dim SU(\tilde{m}/2 + 1) \leqslant k + \dim SU(\bar{m}/2 + 1),$$

or G contains a normal factor $G_1 \approx \mathrm{Spin}(n)$, $n - 1 = t_1 \leqslant \beta = \max(\bar{m} - \alpha, \alpha)$. If $\beta = \alpha$, then by Corollary 3.3

$$\dim K \leqslant \langle m - q - \alpha \rangle + \langle \alpha \rangle.$$

If $\beta = \bar{m} - \alpha$,

$$\begin{aligned} \dim G &\leqslant q + \langle m - q - (\bar{m} - \alpha) \rangle + \langle \bar{m} - \alpha \rangle \\ &= q + \langle k + \alpha - q \rangle + \langle \bar{m} - \alpha \rangle \leqslant k + \langle \alpha \rangle + \langle \bar{m} - \alpha \rangle. \end{aligned}$$

COROLLARY 3.6. (i) $N_T^s(T^k \times CP^n) = \dim SU(n + 1)$.

(ii) $N_T(T^k \times CP^n) = k + \dim SU(n + 1)$.

PROOF. Notice that there is a natural action of $T^k \times SU(n + 1)$ on $T^k \times CP^n$. Hence the results follow from Theorems C and D.

COROLLARY 3.7. *Let $M_i^{m_i}$, $i = 1, 2$, be compact connected topological m_i-manifolds, $m_1 \geqslant m_2$. Then*

(i) $N_T^s(T^k \times M_1^{m_1} \times M_2^{m_2}) \leqslant \langle m_1 \rangle + \langle m_2 \rangle$ *if* $m_1 + m_2 \geqslant 19$.

(ii) *If* $(m_1 - m_2)^2 \geqslant 2(m_1 + m_2)$ *and* $m_1 \geqslant 19$, *then*

$$N_T\big(T^k \times M_1^{m_1} \times M_2^{m_2}\big) \leqslant k + \langle m_1 \rangle + \langle m_2 \rangle.$$

REMARK. We can modify part of the proof of Theorem D to give a different proof of Theorem B(ii) so that we can avoid the use of the gap theorem.

THEOREM 3.8. *Let M^m be a compact connected differentiable m-dimensional A_k-manifold, $k \neq m - 3, m - 1, m$. Then $N^s(M^m) = \langle m - k \rangle$ if and only if M^m is diffeomorphic to $S^{m-k} \times_{Z_2} P$.*

PROOF. By hypothesis, if $\dim G = N^s(M^m)$, then $G \approx \mathrm{Spin}(m - k + 1)$ and the principal orbit is either S^{m-k} or RP^{m-k}. Hence

$$M = \partial\left(D^{m-k+1} \times_{Z_2} P\right),$$

with $M/G \approx P/Z_2$ [17]. It is easy to see that P is an A_k-manifold. But $\dim P = k$; hence $\partial P = \varnothing$. It follows that $M = S^{m-k} \times_{Z_2} P$.

COROLLARY 3.9. *Let M^m be a compact connected differentiable m-dimensional A_{m-2}-manifold. Suppose there exists $u \in H^2(M; Q)$ such that $w_1 \cup \cdots \cup w_{m-2} \cup u \neq 0$.*

(i) *If M^m is not diffeomorphic to $S^2 \times_{Z_2} P$ for any A_{m-2}-manifold P of dimension $m - 2$, then $N^s(M) = 0$.*

(ii) *If $H^*(M; Q) \neq H^*(S^2 \times P; Q)$ for any manifold P, then $N(M^m) \leqslant m - 2$.*

PROOF. It suffices to prove (ii). Suppose $N(M^m) \geqslant m - 1$. Then $N(M^m) = m - 1$. By (i) there is an effective action of T^{m-1} on M^m. Hence the principal orbit is of codimension 1. By a result of Mostert [21] one can show that $M = T^{m-2} \times P$, $P = T^2, RP^2, S^2$ or Klein Bottle, which is a contradiction.

The result (i) is a slight improvement of [5, Theorem B].

Define the *torus-degree of symmetry* $T_t(M)$ of a connected topological manifold M as the maximum of the dimension of torus groups which act effectively on M.

PROPOSITION 3.10. *Suppose M is a compact connected topological m-dimensional A_k-manifold. If the Euler characteristic $\chi(M)$ is nonzero, then $T_t(M) < m - k$. Thus, if $N_T^s(M) = 0$, then $N_T(M) < m - k$.*

PROOF. Let T^n act effectively on M with $T_t(M) = n$, and F be the fixed point set. Then F is not empty. Otherwise, $\chi(M) = \chi(F) = 0$. By [4], we can show that the projection $\pi: M \to M/T^n$ induces a surjection $\pi^*: H^1(M/T^n; Q) \to H^1(M; Q)$ because F is not empty. Hence $m - n = \dim M/T^n > k$.

Suppose now that both $N_T^s(M)$ and $\chi(M)$ are zero. Is $N_T(M) < k$? We can show that the answer is affirmative if $2k \geqslant m - 1$ by using Theorem A in the smooth category.

COROLLARY 3.11. *Let M be a compact connected topological A_{m-3}-manifold of dimension m. Suppose there exists $u \in H^3(M; Q)$ such that $w_1 \cup \cdots \cup$*

$w_{m-3} \cup u \neq 0$ *and the Euler characteristic* $\chi(M)$ *is odd. Then* $N_T(M) \leqslant 3$.

PROOF. By [5, Theorem C], $N_T^s(M) = 0$. Hence $N_T(M) = T_t(M)$. The result follows from Proposition 3.10.

Now we can construct systematically infinitely many manifolds with very little symmetry as follows:

EXAMPLES. Let M^m be a compact connected orientable differentiable m-manifold.

(i) (Cf. [5], [23].) If $\chi(M) \neq 2$, then $N(M \# T^m) = 0$.

(ii) If $\chi(M) \neq 2$, and M^m is not a rational cohomology m-sphere, then

$$N(M \# T^{m-2} \times S^2) \leqslant 2.$$

(iii) If $\chi(M)$ is odd, then $N(M \# T^{m-3} \times S^3) \leqslant 3$.

Question. Suppose $N(M) = 0$. Is $N^s(T^k \times M) = 0$?

LEMMA 3.12 (SEE PROOF OF [19, THEOREM 1]). *Let G be a compact connected Lie group acting almost effectively on an integral cohomology manifold M. Then the group* $\overline{G}$ *can be decomposed as*

$$\overline{G} = T^q \times H \times K \times V$$

where H, K and V are the direct products of G_i*'s (cf.* (1) *for notation) and K and V act almost freely on a principal* $\overline{G}/T^q$ *orbit* M_0 *with* $\dim K < \dim V$. *Moreover, if we let* $H = G_1 \times \cdots \times G_k$, *there are* t_i, $1 \leqslant i \leqslant v$, *satisfying Lemma* 3.2, *with*

$$\sum_{i=1}^{k} t_i = \dim M_0 - \dim V.$$

PROOF OF THEOREM E. Let $\dim G = N_T^s(M)$, where G is semisimple. The group G acts on M with the dimension of principal orbits less than or equal to k_0. Suppose

$$\dim G > \sum_{i=1}^{s+2} \langle m_i \rangle = \sum_{i=1}^{s} \langle k_{i-1} - k_i \rangle + \langle k_s - k_{s+1} \rangle + \langle k_{s+1} \rangle. \quad (11)$$

It follows from (11) that $\dim G > k_0^2/4 + k_0$. Thus from the proof of [16] there exists a normal factor G_1 of G such that $G_1 = \mathrm{Spin}(n_1)$, $n_1 > k_0/2 + 2$. Let $t_1 = n_1 - 1$. Suppose $t_1 > m_1$. Then $n_1 - 2 \geqslant m_1$. Consider the projection $\pi_1\colon M \to M/G_1$. Since $\pi_1^{-1}(x)$ is a point, RP^{n_1-1} or S^{n_1-1} for each $x \in M/G_1$,

$$\pi_1^*\colon H^i(M/G_1; Q) \xrightarrow{\approx} H^i(M; Q), \qquad i < n_1 - 2.$$

Thus $H^{k+m_1+\cdots+m_{s+1}}(M/G_1; Q) \neq 0$. However,

$$\dim M/G_1 = m - t_1 < k + k_0/2 - 1$$

and $k + m_1 + \cdots + m_{s+1} > k + k_0/2 + 1$ by ($*$) and (11). It follows that

$t_1 \leqslant m_1 = k_0 - k_1$. Now we divide the proof into two cases.

(a) $t_1 < k_0 - k_1 - 1$. Let $t_1 = k_0 - k_1 - u$, $u > 1$. Since $t_1 > k_0/2 + 1$, $1 < u < k_0/2 - k_1 - 1$. By Lemma 3.12, the group $\overline{G}$ can be decomposed as $\overline{G} = H \times K \times V$, where H, K and V are each direct products of G_i's and K and V act almost freely on a principal G-orbit M_0 with $\dim K \leqslant \dim V = w$, $H = G_{j_1} \times \cdots \times G_{j_e}$ acts almost effectively on $M_1 = M_0/V$, $\dim G_{j_i} \leqslant \langle t_{j_i} \rangle$, $1 \leqslant i \leqslant e$, and

$$\sum_{i=1}^{e} t_{j_i} = \dim M_0 - \dim V \leqslant k_0 - w. \tag{12}$$

Since V is either a product of simple groups or identity, $\dim V = w \geqslant 3$ or 0. Clearly, G_1 must be a factor of H. We may assume that $G_1 = G_{j_1}$, and denote j_i simply by i, $1 \leqslant i \leqslant e$. Thus we have

$$\sum_{i=2}^{k} t_i \leqslant k_0 - w - t_1 = k_1 + u - w. \tag{13}$$

By Lemma 3.5, since $1 \leqslant u < k_0/2 - k_1 - 1$, we have

$$\langle k_0 - k_1 - u \rangle + \langle k_1 + u \rangle \leqslant \langle k_0 - k_1 \rangle + \langle k_1 - \Phi(k_1) \rangle.$$

It follows that

$$\begin{aligned}
\dim G &= \dim H + \dim V + \dim K \\
&\leqslant \sum_{i=1}^{e} \langle t_i \rangle + 2w \leqslant \langle k_0 - k_1 - u \rangle + \sum_{i=2}^{e} \langle t_i \rangle + 2w \quad \text{by (13)} \\
&\leqslant \langle k_0 - k_1 - u \rangle + \langle k_1 + u - w \rangle + 2w \\
&\leqslant \langle k_0 - k_1 - u \rangle + \langle k_1 + u \rangle - \langle w \rangle + 2w \\
&\leqslant \langle k_0 - k_1 \rangle + \langle k_1 - \Phi(k_1) \rangle - \langle w \rangle + 2w \\
&\leqslant \langle k_0 - k_1 \rangle + \langle k_1 - \Phi(k_1) \rangle \\
&\leqslant \langle k_0 - k_1 \rangle + \langle k_1 - k_2 \rangle \leqslant \sum_{i=1}^{s+1} \langle k_{i-1} - k_i \rangle + \langle k_{s+1} \rangle
\end{aligned}$$

which contradicts (11). Hence $t_1 < k_0 - k_1 - 1$ is impossible. Thus we have

(b) $$t_1 = m_1 = k_0 - k_1.$$

Let (L_1) denote the conjugacy class of the principal isotropy subgroup of the action of $\overline{G}$. The group $K_1 = \overline{G}/G_1$ acts almost effectively on $W_1 = M_{(L_1)}/G_1$ by Lemma 3.1, where

$$M_{(L_1)} = \{x \in M \mid (G_x) = (L_1)\},$$

and $\dim W_1 = m - m_1 = k_1 + k$. It is easy to see that $\dim K_1(x) \leqslant k_1$ for a principal K-orbit $K(x)$, so

$$\dim K_1 > \sum_{i=2}^{s+1} \langle k_{i-1} - k_i \rangle + \langle k_{s+1} \rangle > k_1^2/4 + k_1.$$

Thus, K_1 (and hence $\overline{G}$) contains a normal factor $G_2 = \mathrm{Spin}(n_2)$, $n_2 > k_1/2 + 2$. Let $t_2 = n_2 - 1$. Note that $(M_{(L_1)}/G_1)/G_2 = M_{(L_1)}/G_1 \times G_2$, and the dimension of the principal orbit of $G_1 \times G_2$ action is equal to $t_1 + t_2$. Hence, the identity component of the principal isotropy subgroup is locally isomorphic to $\mathrm{SO}(n_1 - 1) \times \mathrm{SO}(n_2 - 1)$. We can consider the projection π_2: $M \to M/G_1 \times G_2$ and repeat the proof of (a) to show that $t_2 = m_2 = k_1 - k_2$.

We continue the above process by considering $K_2 = \overline{G}/G_1 \times G_2$ acting almost effectively on $W_2 = (W_1)_{(L_2)}/G_2$, where (L_2) is the conjugacy class of the principal isotropy subgroup of K_1 on W_1, $\dim W_2 = k_2$, and so on until we have exhausted $G_1, G_2, \ldots, G_d$, where $d = e$ if $e \leqslant s$, and $d = s + 1$ if $e \geqslant s + 1$. Moreover, for $1 \leqslant j \leqslant d$, $G_j = \mathrm{Spin}(m_j + 1)$, $t_j = m_j = k_{j-1} - k_j$, and $K_j = \overline{G}/G_1 \times \cdots \times G_j$ acts almost effectively on $W_j = (W_{j-1})_{(L_j)}/G_j$ of dimension k_j. There are two subcases.

Subcase (a′). $e \leqslant s$; hence $d = e$. In this case, $\dim H = \sum_{i=1}^{e} \langle k_{i-1} - k_i \rangle$, and

$$\sum_{i=1}^{e} t_i = \sum_{i=1}^{e} (k_{i-1} - k_i) = k_0 - k_e \leqslant m - k - w$$

by (12). Hence $w \leqslant k_e$. As $e \leqslant s$ and $k_e \geqslant 19$,

$$2w \leqslant 2k_e < \langle k_e - \Phi(k_e) \rangle \leqslant \langle k_e - k_{e+1} \rangle.$$

It follows that

$$\dim G \leqslant \dim H + 2w < \sum_{i=1}^{e+1} \langle k_{i-1} - k_i \rangle < \sum_{i=1}^{s+2} \langle m_i \rangle.$$

Subcase (b′). $e \geqslant s + 1$; hence $d = s + 1$. Now K_s acts almost effectively on k_s-manifold W_s and

$$\dim K_s = \dim \overline{G} - \dim G_1 \times \cdots \times G_s > \langle m_{s+1} \rangle + \langle m_{s+2} \rangle \geqslant k_s^2/4 + k_s/2.$$

Hence we must be in one of the possibilities (α) or (δ) of Lemma 3.4. In case of possibility (δ), K_s contains a normal factor $G_{s+1} = \mathrm{Spin}(n_{s+1})$, $n_{s+1} \geqslant k_s/2 + 1$. Then $t_{s+1} = n_{s+1} - 1 \leqslant m_{s+1}$. If not, we get a contradiction by applying the Vietoris-Begle mapping theorem to the projection

$$\pi_{s+1}: M \to M/G_1 \times \cdots \times G_{s+1}.$$

Now we consider the action of K_s on W_s. It follows from Corollary 3.3 that

$$\dim K_s \leqslant \langle m_{s+1} \rangle + \langle k_s - m_{s+1} \rangle = \langle m_{s+1} \rangle + \langle m_{s+2} \rangle.$$

Hence

$$\dim G \leqslant \sum_{i=1}^{s+2} \langle m_i \rangle.$$

If we have the possibility (α), then $W_s = CP^{k_s/2}$, $K_s \approx \mathrm{SU}(k_s/2 + 1)$, and K acts transitively on W_s. Hence $G_s \times K_s$ acts transitively on W_{s-1}, and so on. Finally, $\bar{G}$ acts transitively on M. Moreover, $W_j = M_0/G_1 \times \cdots \times G_j$ for $1 \leqslant j \leqslant s$. In particular,

$$\dim G = \dim G_1 \times \cdots \times G_s \times \mathrm{SU}(k_s/2 + 1)$$
$$= \sum_{i=1}^{s} \langle m_i \rangle + \dim \mathrm{SU}((m_{s+1} + m_{s+2})/2 + 1).$$

This implies the possibility (ii).

COROLLARY 3.13. *If $m_1, m_2, \ldots, m_{s+2}$ satisfy ($*$), then*

$$N_T^s\left(T^k \times M_1^{m_1} \times \cdots \times M_{s+2}^{m_{s+2}}\right) \leqslant \sum_{i=1}^{s+2} \langle m_i \rangle,$$

where $M_i^{m_i}$ are compact connected topological m_i-manifolds.

PROOF OF THEOREM F. The proof will be by induction on s. The assertion is certainly true when $s = 0$ which is precisely Theorem D. If $s > 0$, then we assume by induction that the assertion is true for $s - 1$. Let $\dim G = N(M)$, where $G = (T^q \times K)/N$, K semisimple. If $q \leqslant k$, the result follows from Theorem E. Suppose now that $q > k$ and

$$\dim G > k + \sum_{i=1}^{s+2} \langle m_i \rangle = k + \sum_{i=1}^{s+1} \langle k_{i-1} - k_i \rangle + \langle k_{s+1} \rangle.$$

We divide the proof into two cases.

(a) $q > k_1 + k$. Then $m - q \leqslant k_0 - k_1 - 1$ and

$$\dim G \leqslant q + \langle m - q \rangle \leqslant q + \langle k_0 - k_1 - 1 \rangle$$
$$= k + \langle k_0 - k_1 \rangle + (q - m + k_1) \leqslant k + \sum_{i=1}^{s+2} \langle m_i \rangle$$

because $k_1 \leqslant \sum_{i=2}^{s+2} \langle m_i \rangle$.

(b) $q \leqslant k_1 + k$. Then

$$\dim K > \sum_{i=1}^{s+1} \langle k_{i-1} - k_i \rangle + \langle k_{s+1} \rangle - k_1 > \langle k_0 - k_1 \rangle > k_0^2/4 + k_0$$

by (1). We can repeat the argument of the proof of Theorem E to show that K (and hence G) contains a normal factor $G_1 \approx \mathrm{Spin}(m_1 + 1)$ with possible orbits some combination of fixed points, RP^{m_1} and S^{m_1}. Hence we have

$$M \approx \partial\left(D^{m_1+1} \times_{Z_2} P\right)$$

where $\partial P = F(\mathrm{Spin}(m_1 + 1), M)$. The projection π: $M \to M/G_1$ induces the isomorphism

$$\pi^*: H^i(M/G_1; Q) \to H^i(M; Q) \tag{14}$$

for $i \leqslant m_1 - 1$. Since $M/G_1 \approx P/Z_2$, we have $H^{m-m_1}(P/Z_2, Q) \neq 0$ and $\dim P/Z_2 = \dim M/G_1 = m - m_1$. It follows that $\partial P = \varnothing$ and $M = S^{m_1} \times_{Z_2} P$.

From (14), we see that there exist $\bar{w}_i \in H^1(P; Q)$, $1 \leqslant i \leqslant k$, and $\bar{x}_i \in H^{m_i}(P; Q)$, $2 \leqslant i \leqslant s + 2$, such that

$$\prod_{i=1}^{k} \bar{w}_i \cup \prod_{j=2}^{s+2} \bar{x}_j \neq 0.$$

Moreover, by [13] we can lift the action of G on M to $\bar{G}$ on $\partial(D^{m_1+1} \times P)$. Thus by Lemma 3.1 $\bar{G}/G_1$ acts almost effectively on P [13]. Hence by inductive hypotheses,

$$\dim \bar{G}/G_1 \leqslant k + \sum_{i=2}^{s+2} \langle m_i \rangle,$$

or

$$\dim \bar{G}/G_1 \leqslant k + \sum_{i=2}^{s} \langle m_i \rangle + \dim \mathrm{SU}((m_{s+1} + m_{s+2})/2 + 1).$$

This completes the proof of the theorem.

COROLLARY 3.14. *Let $m_1, m_2, \ldots, m_{s+2}$ satisfy* ($*$). *Then*

$$N(T^k \times M_1^{m_1} \times \cdots \times M_s^{m_s} \times CP^{k_s/2}) = k + \sum_{i=1}^{s} \langle m_i \rangle + N(CP^{k_s/2})$$

where $M_i^{m_i}$ is either diffeomorphic to S^{m_i} or RP^{m_i}.

Suppose now that M^m is an A_k-manifold. If M admits a nontrivial differentiable S^1-action and $i^*: H^1_{S^1}(M; Q) \to H^1(M; Q)$ is not onto, then the action of S^1 cannot extend to S^3-action. It follows that we only need to consider those actions of S^1 on M with $\bar{w}_i \in H^1_{S^1}(M; Q)$, $i^*\bar{w}_i = w_i$, $1 \leqslant i \leqslant k$, in studying $N^s(M)$. By Proposition 2.4 we have

PROPOSITION 3.15. *If for all nontrivial differentiable S^1 actions on M (where $\bar{w}_i$, $1 \leqslant i \leqslant k$, are defined)*

$$\mathrm{Ind}(\bar{w}_1 \cup \cdots \cup \bar{w}_k \cup K_{S^1}(M)) \notin Q[t^2],$$

for some $K_{S^1}(M)$ (see §2 for notation), then $N^s(M) = 0$.

4. Gaps in the dimensions of transformation groups. In this section we shall apply the technique used in the previous sections to obtain the gaps in the dimensions of transformation groups on A_k-manifolds M^m.

THEOREM 4.1. *Let G be a compact connected Lie group acting effectively on a connected topological m-dimensional A_k-manifold M, $\bar{m} \geqslant 17$. If $q \geqslant k$, then* dim G *cannot fall into any of the following ranges*:

$$k + \langle \bar{m} - \alpha \rangle + \langle \alpha \rangle < \dim G \langle k + < \bar{m} - \alpha + 1 \rangle,$$

$$\alpha = 1, 2, \ldots, \Phi(\bar{m}). \quad (15)$$

PROOF. If $q = k$, then (15) reduces to

$$\langle \bar{m} - \alpha \rangle + \langle \alpha \rangle < \dim K < \langle \bar{m} - \alpha + 1 \rangle. \quad (16)$$

But the dimension of K cannot fall into the range (16). Here is an easy proof. Suppose (16) holds. Then

$$\dim K > \bar{m}^2/4 + \bar{m}$$

by (1). According to Lemma 3.4, K contains a normal factor $G_1 = \mathrm{Spin}(n)$, and

$$\dim G_1 = \dim \mathrm{Spin}(n) = \langle n - 1 \rangle = \langle t_1 \rangle < \langle \bar{m} - \alpha + 1 \rangle.$$

Hence $t_1 \leqslant \bar{m} - \alpha$. However, $t_1 > \bar{m}/2 + 2$; hence $t_j \leqslant t_1$, $2 \leqslant j \leqslant v$. Thus

$$\dim K \leqslant \langle \bar{m} - \alpha \rangle + \langle \alpha \rangle$$

by Corollary 3.3.

Suppose now that $q > k$. If (15) holds, then

$$\dim G > k + \langle \bar{m} - \alpha \rangle + \langle \alpha \rangle.$$

We can use the same proof as the proof of Theorem D to show that

$$\dim K > \langle m - q - \alpha \rangle + \langle \alpha \rangle.$$

If $n - 1 \leqslant \bar{m} - \alpha$, we get

$$\dim G \leqslant k + \langle \bar{m} - \alpha \rangle + \langle \alpha \rangle$$

by Corollary 3.3 which is a contradiction. Hence $\dim \mathrm{Spin}(n) \geqslant \langle \bar{m} - \alpha + 1 \rangle$ and

$$\dim G \geqslant q + \langle \bar{m} - \alpha + 1 \rangle > k + \langle \bar{m} - \alpha + 1 \rangle$$

which contradicts (15).

The assumption $q \geqslant k$ is necessary, as we can see from the following example. Let $M = T^k \times S^{\bar{m}}$ and $G = T^q \times \mathrm{Spin}(\bar{m})$ and G acts diagonally on M, $q < k$. Then

$$k + \langle \bar{m} - 2 \rangle + \langle 2 \rangle < q + \langle \bar{m} - 1 \rangle = \dim G < k + \langle \bar{m} - 1 \rangle$$

if $2k + 4 \leqslant m - q$.

REMARK 4.2. We have observed in [**14**] that Lemma 3.2, Corollary 3.3 and Lemma 3.4 remain true for integral cohomology manifolds. Hence Theorem 4.1 can be stated for integral cohomology manifolds. The boundary is not necessarily empty because we only use the fact that for effective semisimple Lie group actions on A_k-manifolds, the principal orbits have dimension less than or equal to $\bar{m} = m - k$.

LEMMA 4.3 [14], [20]. *Let M be a connected integral cohomology m-manifold and k_i $(i = 0, 1, \ldots, s+1)$ any sequence of positive integers satisfying $k_0 = m$, $k_{i+1} \leqslant \Phi(k_i)$, $0 \leqslant i \leqslant s$, and $k_s \geqslant 17$. If G is a compact connected Lie group acting effectively on M, then dimension of G cannot fall into any of the following ranges:*

$$\sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} \rangle + \langle k_{s+1} \rangle < \dim G$$
$$< \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} + 1 \rangle. \tag{17}$$

PROOF. We shall give a new simple proof (cf. [14]). The proof will be by induction on s. The assertion is true when $s = 0$ by Remark 4.2 (take $k = 0$). If $s > 0$, then we assume by induction that the assertion is true for $s - 1$. Let the Lie group G satisfy (17). Then we have

$$\dim G > \langle k_0 - k_1 \rangle > m^2/4 + m.$$

By Lemma 3.4, there is a normal factor G_1 of G such that $G_1 \approx \mathrm{Spin}(n_1)$, $n_1 > m/2 + 2$. Let $t_1 = n_1 - 1$. If $\dim G_1 \geqslant \langle k_0 - k_1 + 1 \rangle$, then

$$\sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} + 1 \rangle$$
$$\leqslant \langle k_0 - k_1 \rangle + \left\langle \sum_{i=1}^{s-1} (k_i - k_{i+1}) + k_s - k_{s+1} + 1 \right\rangle$$
$$\leqslant \langle k_0 - k_1 \rangle + \langle k_1 - k_{s+1} + 1 \rangle \leqslant \langle k_0 - k_1 \rangle + \langle k_1 \rangle$$
$$\leqslant \langle k_0 - k_1 + 1 \rangle \leqslant \dim G_1,$$

which is impossible. Thus $\dim G_1 = \langle t_1 \rangle \leqslant \langle k_0 - k_1 \rangle$, and $t_1 \leqslant k_0 - t_1$. We shall show that $t_1 = k_0 - k_1$. Otherwise $t_1 = k_0 - k_1 - u$, $u \geqslant 1$. By Lemma 3.12, we have the decomposition $G = T^q \times H \times V \times K$. We may assume that $H = G_1 \times \cdots \times G_k$. Moreover

$$\sum_{i=2}^{k} t_i = \dim M_0 - \dim V - t_1 \leqslant k_1 + u - q - w,$$

where $w = \dim V$. Notice that $t_1 > k_0/2 + 1$, hence $1 \leqslant u \leqslant k_0/2 - k_1 - 1$. Thus we can apply Lemma 3.5. It follows that

$$\begin{aligned}\dim G &= \dim H + \dim V + \dim K + q \leqslant \sum_{i=1}^{k} \langle t_i \rangle + 2w + q \\ &= \langle k_0 - k_1 - u \rangle + \sum_{i=2}^{k} \langle t_i \rangle + 2w + q \\ &\leqslant \langle k_0 - k_1 - u \rangle + \left\langle \sum_{i=2}^{k} t_i \right\rangle + 2w + q \\ &\leqslant \langle k_0 - k_1 - u \rangle + \langle k_1 + u - q - w \rangle + 2w + q \\ &\leqslant \langle k_0 - k_1 - u \rangle + \langle k_1 + u \rangle - \langle q + w \rangle + 2w + q \\ &\leqslant \langle k_0 - k_1 \rangle + \langle k_1 - \Phi(k_1) \rangle - \langle q + w \rangle + 2w + q \\ &\leqslant \langle k_0 - k_1 \rangle + \langle k_1 - \Phi(k_1) \rangle \leqslant \langle k_0 - k_1 \rangle + \langle k_1 - k_2 \rangle\end{aligned}$$

which contradicts (17). Hence $t_1 = k_0 - k_1$. Since the G_1-orbits are some combination of S^{t_1}, RP^{t_1} and fixed points, by Lemma 3.1, the group $K_1 = \overline{G}/G_1$ acts almost effectively on $M_{(H)}/G_1$, where (H) denotes the conjugacy class of the principal isotropy subgroup of the action of $\overline{G}$. Now the dimension of $M_{(H)}/G_1$ is equal to $k_0 - t_1 = k_1$, by inductive hypothesis, $\dim K_1$ cannot fall into the following ranges:

$$\sum_{i=1}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} \rangle + \langle k_{s+1} \rangle < \dim K_1 < \sum_{i=1}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} + 1 \rangle.$$

It follows that $\dim G$ cannot fall into the ranges (17).

PROOF OF THEOREM G. By Lemma 4.3, $\dim K$ cannot fall into the range (17). Hence, $q > k$. Suppose now that

$$k + \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} \rangle + \langle k_{s+1} \rangle < \dim G < k + \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} + 1 \rangle$$

and $q > k$. We consider two cases.

(a) $q > k_1 + k$. This is impossible (see proof of Theorem F).

(b) $q \leqslant k_1 + k$. Then we can show as before that G contains a normal factor $G_1 \approx \mathrm{Spin}(m_1 + 1)$. Moreover, the orbit space M/G_1 is an integral cohomology A_k-manifold possibly with boundary of dimension $m - m_1$. The group $K_1 = \overline{G}/G_1$ acts almost effectively on M/G_1. Since the theorem is true for $s = 0$, by induction $\dim K_1$ cannot fall into the following range:

$$k + \sum_{i=1}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} \rangle + \langle k_{s+1} \rangle$$
$$< \dim K_1 < k + \sum_{i=1}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} + 1 \rangle.$$

This contradicts (18), and the proof of the theorem is complete.

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